MTH849 QUALIFICATION EXAM (MAY 4, 2023)

Instructions.

- You will have **120 minutes** to complete this exam.
- No outside resources of any kind are allowed.
- Write solutions legibly and in an organized manner.
- Write your solution to each question on a separate answer sheet (you may use multiple sheets per question), label each sheet with your qual number, question number, sheet number, and total number of sheets used. Do not write on backs of sheets.
- Problems will be graded according to the holistic grading scheme as detailed on the back of this page.

Unless otherwise specified, $\Omega \subset \mathbb{R}^n$ is a bounded, open set with C^1 boundary.

Problem 1. Fix $0 < p < \infty$ and let us write (x, y) for points in $\mathbb{R}^2$. Suppose that for some $1 \le q < \infty$ we want to prove an inequality of the form

$$\|u\|_{L^q(\mathbb{R}^2)} \le C \left(\int_{\mathbb{R}^2} \left| \frac{\partial^3 u(x,y)}{\partial x^2 \partial y} \right|^p + \left| \frac{\partial^2 u(x,y)}{\partial y^2} \right|^p dxdy \right)^{\frac{1}{p}}, \ \forall\ u \in C_c^3(\mathbb{R}^2)$$

where $C > 0$ is independent of u. Using a scaling argument, prove there is only one possible value of q for each p such that the above inequality can hold, and find this value. You do not need to prove the inequality holds for that value.

Problem 2. Fix an arbitrary function $\eta \in L^q(\Omega)$ for some $1 \le q$. Prove that if $1 < n < q$, then $\int_\Omega \eta^2 |u||v| < \infty$ for all $(u, v) \in W^{1,\frac{n}{n-1}}(\Omega) \times W^{1,n}(\Omega)$.

Problem 3. Let $Lu := -\partial_j(\delta^{ij}\partial_i u) - u$ where $\delta^{ij} = 1$ when $i = j$ and 0 when $i \ne j$. Prove that if $\{u_k\}_{k=1}^\infty \subset H_0^1(\Omega)$ is a sequence of weak solutions to

$$Lu = 1 \text{ on } \Omega,\ \ u \equiv 0 \text{ on } \partial\Omega$$

and $\sup_k \|u_k\|_{L^2(\Omega)} < \infty$, then there is a subsequence which converges in $\boldsymbol{H^1(\Omega)}$ norm.

Problem 4. Prove that if $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ are functions satisfying

$$-\sum_{i=1}^n (1 + |\partial_i u|)^3 \partial_{ii}^2 u + \arctan u \le -\sum_{i=1}^n (1 + |\partial_i v|)^2 \partial_{ii}^2 v + \arctan v \text{ on } \Omega, \quad u \le v \text{ on } \partial\Omega$$

then $u \le v$ on all of Ω.

Problem 5. Let Ω be the open rectangle $(0, 2) \times (1, 4) \subset \mathbb{R}^2$, $(a^{ij}) = \begin{pmatrix} \frac{9}{2} - y & 0 \\ 0 & 1 \end{pmatrix}$, and $Lu = -\partial_j(a^{ij}\partial_i u)$. Using Rayleigh's formula, give a nontrivial lower bound on the principal Dirichlet eigenvalue for the eigenvalue problem

$$Lu = \lambda u \text{ on } \Omega, \quad u \equiv 0 \text{ on } \partial\Omega.$$

You may give any lower bound if it is **fully justified**, but the bound must be **strictly positive** and **an explicit number** to receive credit.

Question	Holistic grade	Points
Q1		
Q2		
Q3		
Q4		
Total		

Grading scale

- **E**xcellent: essentially flawless, 100% of grade.
- **G**ood: minor mistakes, but generally correct, 90% of grade.
- **F**air: missing some key steps or arguments, but on the right track, 70% of grade.
- **W**rong approach: uses a not-promising approach to the problem but concludes the approach in a correct manner to its logical conclusion, 50% of grade.
- **A**ttempted: an honest but not satisfactory attempt, 30% of grade.
- **N**o attempt: no work or mostly nonsense solution, 0% of grade.